AP Calculus AB
Lesson 5-5: *Linearization & Differentials*, Part 1

Name Hei... 2016
Date

Learning Goal:

- *I can find the linearization of a function at* $x = a$.

Early in our study of derivatives, we made a statement that if a function is differentiable at a point, then it is "locally linear" at that point. We explored this fact a bit by zooming in on two graphs that looked similar. The two functions were $f_1(x) = |x| + 1$ and $f_2(x) = \sqrt{x^2 + 0.0001} + 0.99$. The graph of f_1 had a corner no matter how far you zoomed in, and the graph of f_2 eventually looked linear when zoomed in. In this section, we are going to explore the idea of local linearity a bit more.

Exploration – Appreciating Local Linearity

1. Graph the function $f(x) = (x^2 + 0.0001)^{\frac{1}{4}} + 0.9$ in "Zoom Decimal" mode. What appears to be the behavior of the function at $(0,1)$? What could this behavior mean?

   There appears to be a corner, so it would not be differentiable at $x = 0$.

2. If you use the definition of a derivative, you can prove that f is in fact differentiable at the point $(0,1)$. Since f is differentiable, let's find the derivative where there appears to be a corner, at $x = 0$. By hand (just for practice) find $f'(0)$ below.

   $f'(x) = \frac{1}{4}(x^2 + .0001)^{-\frac{3}{4}} \cdot 2x = \frac{1}{2}x(x^2 + .0001)^{-3/4}$

   $f'(0) = 0$

3. What is the equation of the tangent line to f at $x = 0$. Remember, to find the equation of a tangent line you need the slope of the tangent line (problem 2) and a point on the graph (also in problem 2). Then find the equation (point-slope form is the easiest).

   $y - 1 = 0(x - 0)$ point: $(0, 1)$

   $y - 1 = 0$ slope: 0

   $\boxed{y = 1}$

4. Graph the tangent line in the same window as the graph of f. Does the tangent line appear to approximate the graph of f at $x = 0$?

   Not really

5. Zoom in repeatedly, centered at $(0,1)$. You will probably need to zoom in at least 10-20 times. What happens to the graph and the tangent line?

   They appear to be the same when enough zoom is applied.

OVER →

DEFINITION – Linearization

If f is differentiable at $x = a$, then the equation of the tangent line

condition

$$L(x) = f(a) + f'(a)(x-a)$$

★ just a variation of pt.-slope form!

Defines the **linearization approximation of f at a.** The point $x = a$ is the **center** of the approximation.

* *Note that this is just the equation of the tangent line to f at a. It is just point slope form rewritten with the y-value on the other side of the equation!*

Example

$(1+x)^{\frac{1}{2}}$

Find the linear approximation of $f(x) = \sqrt{1+x}$ at $x = 0$, and use it to approximate $\sqrt{1.02}$ without a calculator. Check the accuracy of your estimate on your calculator.

$f(0) = 1$

$(0, 1)$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f'(0) = \frac{1}{2} \cdot 1$

$= \frac{1}{2}$

$L(x) = 1 + \frac{1}{2}(x-0)$

$= 1 + \frac{1}{2}x$ use this

To approximate $\sqrt{1.02}$, use $x = .02$

$\sqrt{1.02} = f(.02)$

$\approx L(.02)$

$= 1 + \frac{.02}{2} = 1.01$

Calc: ≈ 1.00995

(note that in general, for $k \neq 0$, for function in the form $(1+x)^k$, $L(x) = 1 + kx$)

Practice:

1. Find the linear approximation of $f(x) = \cos x$ at $x = \frac{\pi}{2}$ and use it to approximate $\cos(1.75)$ without a calculator. **Explain what the value of $\cos(1.75)$ means in the context of this problem.**

$f'(x) = -\sin x$ $\quad f(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$(\frac{\pi}{2}, 0)$

$f'(\frac{\pi}{2}) = -1$

$\cos(1.75) = f(1.75) \approx L(1.75)$

$L(x) = 0 + -1(x - \frac{\pi}{2})$

$= -x + \frac{\pi}{2}$

$= -1.75 + \frac{\pi}{2}$

★ At $(\frac{\pi}{2}, 0)$ the line $y = -x + \frac{\pi}{2}$ approximates the graph of $f(x) = \cos x$.

$\approx -1.75 + 1.57$

$= -.18$ $\quad$ Calc.: $\approx -.178246$

2. Find the linear approximation of $y = x + \frac{1}{x}$ at $x = 1$. Explain what your answer tells you about the function.

$f(1) = 2$

$f'(x) = 1 - \frac{1}{x^2}$

$f'(1) = 0$

$L(x) = 2 + 0(x-1)$

$L(x) = 2$

At $(1, 2)$ the line $y = 2$ approximates the graph $y = x + \frac{1}{x}$.

OVER →

Practice (continued)

3. Find the linear approximation of $\sqrt{68}$. Use the closest perfect square to 68 for your point of tangency.

$f(x)=\sqrt{x}$

$f(64)=8 \rightarrow (64,8)$

$f'(x)=\frac{1}{2\sqrt{x}} \rightarrow f'(64)=\frac{1}{16}$

$L(x)=8+\frac{1}{16}(x-64)$

$L(68)=8+\frac{1}{16}(68-64)$

$L(68)=8.25$

calc: 8.2462

☆ Note; we are finding the equation of the T.L. at $x=64$ & using it to approximate $f(x)=\sqrt{x}$. ☺

4. *NO CALCULATOR*

The best linear approximation for $f(x)=\tan x$ near $x=\frac{\pi}{4}$ is

(A) $1+\frac{1}{2}\left(x-\frac{\pi}{4}\right)$ (B) $1+\left(x-\frac{\pi}{4}\right)$ (C) $1+\sqrt{2}\left(x-\frac{\pi}{4}\right)$

(D) $1+2\left(x-\frac{\pi}{4}\right)$ (E) $2+2\left(x-\frac{\pi}{4}\right)$

$f\left(\frac{\pi}{4}\right)=1 \quad \left(\frac{\pi}{4},1\right)$

$f'(x)=\sec^2 x$

$f'\left(\frac{\pi}{4}\right)=\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}=\frac{1}{\frac{1}{2}}=2$

☆ variation of pt.-slope

$y-1=2\left(x-\frac{\pi}{4}\right)$

$\quad +1 \qquad +1$

$L(x)=1+2\left(x-\frac{\pi}{4}\right)$

5. *CALCULATOR ACTIVE*

The approximate value of $y=\sqrt{4+\sin x}$ at $x=0.12$, obtained from the tangent to the graph at $x=0$ is

$=(4+\sin x)^{\frac{1}{2}}$

(A) 2.00 $\boxed{\text{(B) 2.03}}$ (C) 2.06 (D) 2.12 (E) 2.24

$y(0)=\sqrt{4+\sin 0}=2 \quad (0,2)$

$y'=\frac{1}{2}(4+\sin x)^{-\frac{1}{2}}\cdot\cos x$

$y'(0)=\frac{1}{2}(4+0)^{-\frac{1}{2}}\cdot\cos 0$

$=\frac{1}{2}\cdot\frac{1}{2}\cdot 1=\frac{1}{4}$

$L(x)=2+\frac{1}{4}(x-0)$

$=2+\frac{1}{4}x$

$L(.12)=2+\frac{1}{4}(.12)$

$=2+.03$

OVER →

AP Calculus AB — Name: Hent 2016
Lessons 5-5 *Linearization & Differentials*, Part 2 — Date:

Learning Goal:

- *I can find and evaluate a differential.*

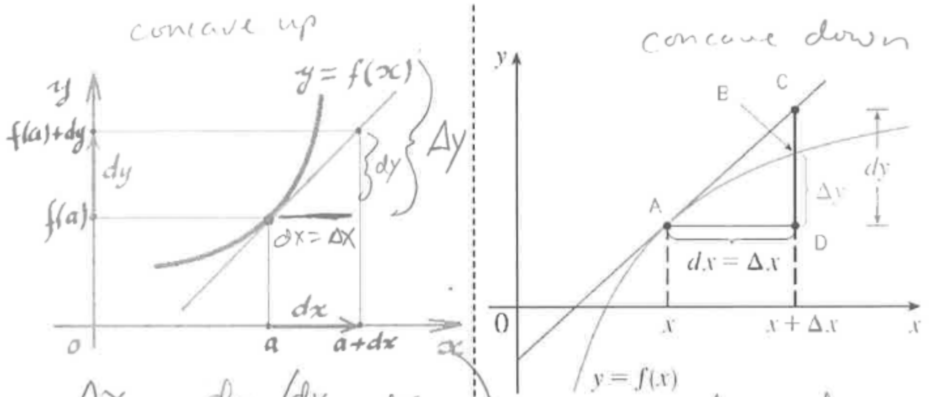
Differentials

DEFINITION: Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy is**

$$dy = f'(x)\,dx$$

Unlike the independent variable dx, the variable dy is always a dependent variable.
It depends on both x and dx.

As you can see below, the linearization of a function f at $x = a$ can sometimes underestimate the change (concave up) and other times overestimate the change (concave down) in the function f.



$\frac{\Delta x}{\Delta y} > \frac{dy}{dx}$ ($\frac{dy}{dx}$ under-estimates the change)

$\frac{\Delta y}{\Delta x} < \frac{dy}{dx}$ ($\frac{dy}{dx}$ overestimates the change)

Example
If $y = x^3 - 2x^2 + 3x - 4$, find the differential dy and evaluate dy when $x = 1$ and $dx = 0.01$

$dx\left(\frac{dy}{dx}\right) = (3x^2 - 4x + 3)dx$

$dy = (3x^3 - 4x + 3)dx$

$dy = (3(1)^2 - 4(1) + 3)(.01)$

$dy = .02$

Practice

In the following practice problems, find the differential *dy* and evaluate *dy* at the given value when

(1) $y = \sin(3x)$; when $x = \pi$ and $dx = -0.02$

(2) $x + y = xy$; when $x = 2$ and $dx = 0.05$

(1) $\frac{dy}{dx} = 3\cos 3x$

$dy = (3\cos 3x)\,dx$

$= 3 \cdot \cos(3\pi) \cdot -.02$

$= 3 \cdot -1 \cdot -.02$

$dy = .06$

(2) $1 + \frac{dy}{dx} = 1 \cdot y + \frac{dy}{dx} \cdot x$

$1 + \frac{dy}{dx} = y + x \cdot \frac{dy}{dx}$

$\frac{dy}{dx} - x \cdot \frac{dy}{dx} = y - 1$

$\frac{dy}{dx}(1 - x) = y - 1$

$dx\left(\frac{dy}{dx}\right) = \left(\frac{y-1}{1-x}\right)dx$

$dy = \left(\frac{y-1}{1-x}\right)dy$

$2 + y = 2y$

$2 = y$

$(2,2)$

$dy = \frac{2-1}{1-2} \cdot .05$

$= -1(.05)$

$= \boxed{-.05}$

Mixed AP Practice – *No Calculator*

2. If $f(x) = e^{2x}(x^3 + 1)$, then $f'(2) =$

(A) $6e^4$ (B) $21e^4$ (C) $24e^4$ (D) $30e^4$

$f(x) = 2e^{2x}(x^3+1) + e^{2x}(3x^2)$

$f'(2) = 2e^{2 \cdot 2}(2^3+1) + e^{2 \cdot 2}(3 \cdot 2^2)$

$= 2e^4 \cdot 9 + e^4 \cdot 12 = 18e^4 + 12e^4 = 30e^4$

OVER →

$f^{-1}: \overset{(a,b)}{(2,0)}$ Page 6

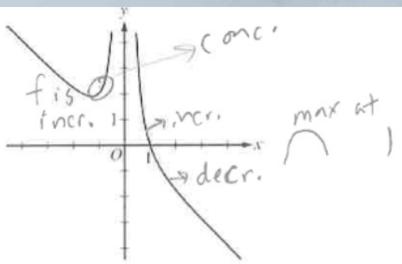
29. The function f is defined by $f(x) = x^3 + 4x + 2$. If g is the inverse function of f and $g(2) = 0$, what is the value of $g'(2)$?

(A) $-\frac{1}{16}$ (B) $-\frac{4}{81}$ (C) $\frac{1}{4}$ (D) 4

$f'(x) = 3x^2 + 4 \Rightarrow f'(0) = 0 + 4 = 4$

$g = f^{-1}$

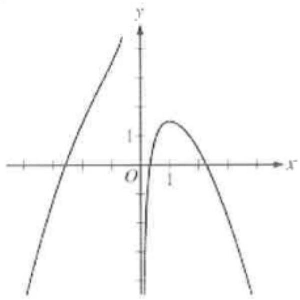
$(f^{-1}(a))' = \frac{1}{f'(b)}$ | $(f^{-1}(2)) = \frac{1}{f'(0)} = \frac{1}{4}$



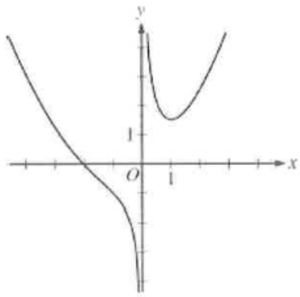
Graph of f'

81. The graph of f', the derivative of the function f, is shown above. Which of the following could be the graph of f?

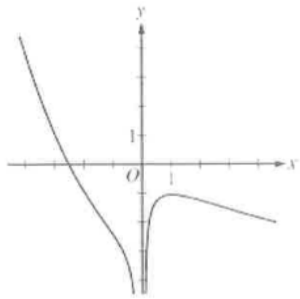
(A)



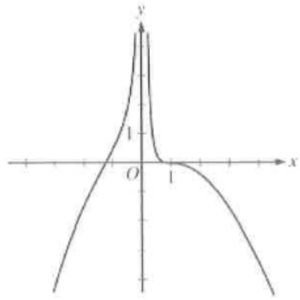
(B)



(C)



(D)



Lesson 5-5 HW Answers

pp. 242–243 / 1, 2, 12, 21

1. $f(x) = x^3 - 2x + 3$, $a = 2$

a. $f(2) = 7$

$f'(x) = 3x^2 - 2$

$f'(2) = 10$

$L(x) = 7 + 10(x-2)$

b. $f(2.1) \approx L(2.1) =$

$7 + 10(0.1) =$

$7 + 1 = \boxed{8}$

2. $f(x) = \sqrt{x^2+9}$, $a = -4$

$= (x^2+9)^{\frac{1}{2}}$

a. $f(-4) = \sqrt{25} = 5$

$f'(x) = \frac{1}{2}(x^2+9)^{-\frac{1}{2}} \cdot 2x$

$= x(x^2+9)^{-\frac{1}{2}}$

$f'(-4) = -4(25)^{-\frac{1}{2}} = -\frac{4}{5}$

$L(x) = 5 + -\frac{4}{5}(x+4)$

b. $f(-3.9) \approx L(-3.9)$

$= 5 + -\frac{4}{5}(-3.9+4)$

$= 5 + -\frac{4}{5}(.1) = 5 + (-.8)(.1)$

$= 5 - .08$

$= 4.92$

12. $\sqrt[3]{26}$ 27 is closest perfect cube.

$f(x) = \sqrt[3]{27-x}$

$f(0) = 3$

$f'(x) = \frac{1}{3}(27-x)^{-\frac{2}{3}} \cdot -1$

$f'(0) = -\frac{1}{27}$

$L(x) = 3 - \frac{1}{27}(x-0)$

$L(1) = 3 - \frac{1}{27} \cdot 1$

$= 3 - \frac{1}{27}$

$\approx 2.\overline{962}$

17. $y = x^2 \ln x$ $x = 1$ $dx = .01$

$\frac{dy}{dx} = 2x \ln x + x^2 \cdot \frac{1}{x}$

$dx\left(\frac{dy}{dx}\right) = (2x \ln x + x)\,dx$

$dy = (2 \cdot 1 \cdot \ln 1 + 1) .01$

$dy = 1 \cdot .01$

$\boxed{dy = .01}$

21. **(a)** $y + xy - x = 0$

$y(1+x) = x$

$y = \frac{x}{x+1}$

Since $\frac{dy}{dx} = \frac{(x+1)(1) - (x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$,

$dy = \frac{dx}{(x+1)^2}$.

(b) At the given values, $dy = \frac{0.01}{(0+1)^2} = 0.01$.